Langmuir oscillations in a cold inhomogeneous plasma

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One-dimensional nonlinear Langmuir oscillations in a cold inhomogeneous plasma are considered. The spatial distribution of the amplitude of oscillations is set by the initial disturbance and can be arbitrary, i.e., it is not in any way connected with the inhomogeneity of the plasma. Taking place at small spatial gradients of the initial disturbance because of a growing with time phase shift of oscillations of different electrons, the effect of formation of narrow density peaks is illustrated. The density peaks always move in the direction of a decrease of ion concentration. In this movement their amplitude can grow or decrease depending on the direction of the initial disturbance gradient. A number of electrons, forming density peaks, reduce with a diminution of plasma inhomogeneity but the peak amplitudes always increase unlimitedly with time up to the self-intersection of electron trajectories. An equation, making it possible to determine the instant of trajectory intersection in case of small plasma inhomogeneity of arbitrary kind and arbitrary gradients of the initial disturbance, is obtained. It is shown by a numerical calculation that the analytical relations obtained are not essentially changed if one takes into account the nonlinearity of corresponding differential equations.

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I. INTRODUCTION

It is known that the exact solution of equations, describing one-dimensional movements of electrons in a cold collisionless homogeneous plasma, is undamped periodic oscillations with basic frequency ω_p that is the Langmuir frequency. This solution is true up to some critical value a_{cr} of the amplitude of oscillations. The spatial structure and the amplitude of oscillations are set by the initial disturbance. The intersection of electron trajectories occurs at amplitudes $a > a_{\mathrm{cr}}$ [1,2], periodic oscillations are destroyed, and a peculiar one-dimensional turbulence, not studied completely yet, develops.

In Ref. [3], dedicated to the $a < a_{\mathrm{cr}}$ case, it is shown that an electric field of Langmuir oscillations has a time-independent component, which inevitably puts ions in motion. Their distribution in space becomes nonuniform and, giving most of their own energy to chaotic electron movement, electron oscillations quickly damp. It seems expedient to learn in more detail every "elementary" process that determines the development of Langmuir turbulence in such a scenario, in particular, peculiarities of Langmuir oscillations of an inhomogeneous plasma.

Considerable clarification on this subject was achieved long ago on a basis of qualitative considerations of small amplitude oscillations. They take place at the local plasma frequency ω_p so that as incoordination of oscillations of individual electron layers increases, the length of the perturbation wave decreases with time [4]. This circumstance leads to the intersection of the trajectories after some time t_c [5] which, in the case of coordinate-independent amplitude, is determined in the following way:

$$t_c \approx \frac{1}{\xi_0(\partial\omega_p(x_0)/(\partial x_0))}\ , \tag{1}$$

where x_0 is the coordinate of electron equilibrium; ξ_0 is the amplitude of oscillation of electron displacement.

Attention has also been paid to the case of wave breaking when analyzing the oscillations, engendered by an abrupt ion density disturbance of one form or another [6–8]. In this case the authors have essentially considered the solution when the spatial distribution of amplitude was connected rigidly with the inhomogeneity of ion concentration. In the general case, the amplitude of oscillations and the inhomogeneity of plasma concentration are independent parameters.

The present paper is dedicated to a detailed analysis of Langmuir oscillations in this general case. Special attention too is paid to visualization of the process.

II. WAVE STRUCTURE IN A WEAKLY INHOMOGENEOUS PLASMA

We consider a cold collisionless one-dimensional plasma, which has a concentration $n_i(x)$ of motionless ions related somehow to a coordinate in some spatial interval (Fig. 1). Electrons are distributed like ions at an initial moment of time ($t=0$) so that the electric field is 0 ($E|_{t=0}=0$). Let us introduce Lagrange's variable x_0 for electron coordinates at time $t=0$ and let us also set the one-dimensional disturbance of electron velocities at the same time $dx/dt|_{t=0}=v(x)$, which will put the electrons in motion. We calculate the change of the electric field, accompanying any chosen electron, if one is shifted from the initial position x_0 in the coordinate x (Fig. 1). Because intersections of electron trajectories do not occur till some time t_c, the electron charge, which is to the left (and accordingly, to the right) of the observed electron does not change and the desired electric field is determined by the ion charge confined in interval $[x_0,x]$. Thus,

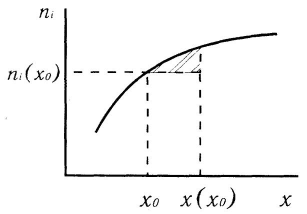

FIG. 1. Condition for the deduction of the equation of electron oscillations in an inhomogeneous plasma.

$$E=4\pi e n_i(x-x_0)+4\pi e S(x,x_0) , \quad (2)$$

where $S(x,x_0)=\int_{x_0}^{x}[n_i(x')-n_i(x_0)]dx'$ is the shaded square in Fig. 1. Using (2), the equation of electron movement can be written as

$$\frac{d^2x}{dt^2}=-\omega_p^2(x_0)(x-x_0)-\omega_p^2(x_0)\frac{S}{n_i(x_0)} , \quad (3)$$

where $\omega_p^2=4\pi e^2 n_i(x_0)/m$ is the local plasma frequency. By introducing new variables such as displacement of the electron from its initial position of equilibrium $\delta=x-x_0$ and dimensionless time $\tau=\omega_p t$, we rewrite (3) in the form of

$$\frac{d^2\delta}{d\tau^2}=-[1+R(x_0,\delta)]\delta , \quad (4)$$

where $R=S/n_i\delta$. Initial conditions for Eq. (4) appear as follows:

$$\delta|_{t=0}=0 , \quad \left.\frac{d\delta}{d\tau}\right|_{t=0}=\frac{v(x_0)}{\omega_p} . \quad (5)$$

In explicit form the approximate analytical solution for Eq. (4) may be found in the case

$$|R|\ll 1 , \quad (6)$$

when this equation is transformed into a linear one by rejection of the nonlinear in the δ term. The solution of the linear equation with the initial conditions (5) gives

$$\delta=\frac{v(x_0)}{\omega_p(x_0)}\sin\tau , \quad (7)$$

or

$$x=x_0+\frac{v}{\omega_p}\sin(\omega_p t) . \quad (8)$$

Let us note that R is the dimensionless parameter of inhomogeneity, which implies relative change of ion concentration on displacement amplitude of the oscillating electron. If the dependence of ion concentration can be presented by the linear function $n_i(x)=n_i(x_0)+k(x-x_0)$ in some spatial interval, then the corresponding parameter of inhomogeneity R_l is expressed by means of the initial conditions as follows:

$$R_l=\left|\frac{kv(x_0)}{2\omega_p(x_0)n_i(x_0)}\right| . \quad (9)$$

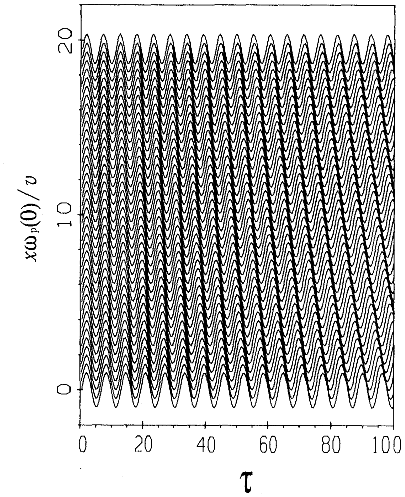

FIG. 2. Condensations of trajectories in a linear inhomogeneous plasma in the case where the initial velocity of electrons does not depend on the coordinate.

For concreteness a linear inhomogeneous plasma with the parameter $R_l \ll 1$ is considered unless otherwise stipulated.

The expression (8) describes the continuum of trajectories with initial conditions and frequencies continuously distributed in x. The calculated trajectories of 40 electron sheets, situated equidistantly on the spatial interval $0<x<20$ at the initial moment of time, are given in Fig. 2. [Time is normalized in the plasma oscillation period at $x=0$ and distance is normalized in the amplitude of displacement of this electron, i.e., on the $v(0)/\omega_p(0)$]. The initial disturbance of electron velocities v is independent of x_0. The parameter of inhomogeneity R_l is 0.01. Condensations of electron trajectories, appearing and growing with time, moving in the direction of the ion density decrease, are well seen in this figure.

A more accurate notion about spatial and temporal dynamics of forming nonlinear waves of electron density may be obtained if the continuity equation for the electron component is used in Lagrange's form,

$$n(x)|_{t=0}\Delta x_0=n(x,t)\Delta x , \quad (10)$$

where Δx_0 is the distance between any two close electron sheets at an initial moment of time, and Δx is a distance between them at time t. Equation (10) simultaneously with (8) allows the distribution of electron density at any time t to be calculated. A series of corresponding dependencies, obtained under the same conditions as in Fig. 2 and under the same normalization of t and x, are shown in Fig. 3. Electron density is normalized in the undisturbed density at $x=0$. The dashed line gives the spatial distribution of ion density.

One may see that in the case under consideration the spatial structure of oscillations has already become complicated at the third period of oscillations. Subsequently, narrow peaks of density form, moving from right to left with simultaneous increase of amplitude. Analogous calculations, made in the present work for the cases when v depends on x, have shown that a decrease in amplitudes of density peaks at their motion is possible as well. However, at any fixed point x, every density peak has a greater amplitude than the preceding one had. The growth of amplitude is not limited and tells us about condensation of trajectories up to their intersection, i.e., formation of "folds" or "overlapping" in an electron medium.

III. SELF-INTERSECTION OF ELECTRON TRAJECTORIES

A. Linear approach

Let us use Eq. (8) for finding the time of the first intersection t_c. Denoting initial coordinates of any two electrons, which are close enough, by x_{01} and x_{02} and their initial velocities by v_1 and v_2 accordingly, we write equations of trajectories of these electrons,

$$x_1 = x_{01} + \frac{v_1}{\omega_{p1}} \sin(\omega_{p1} t) ,$$
$$x_2 = x_{02} + \frac{v_2}{\omega_{p2}} \sin(\omega_{p2} t) , \tag{11}$$

where ω_{p1}, ω_{p2} are local plasma frequencies, corresponding to points x_{01} and x_{02}. x_1 coincides with x_2 at the moment t_c; hence,

$$\frac{(x_{01}-x_{02})\omega_{p1}}{v_1} - \frac{\omega_{p1}}{\omega_{p2}} \frac{v_2}{v_1} \sin(\omega_{p2} t_c) - \sin(\omega_{p1} t_c) . \tag{12}$$

Using the notation $\omega_{p2} - \omega_{p1} = \Delta\omega_p$, we let $\Delta\omega_p t_c \ll 1$. In Eq. (12), in this case, writing $\sin(\omega_{p2} t_c)$ as $\sin(\omega_{p1} t_c + \Delta\omega_p t_c)$ and expanding it into a series in powers of $\Delta\omega_p t_c$, we may retain the linear term only. Thus, we have

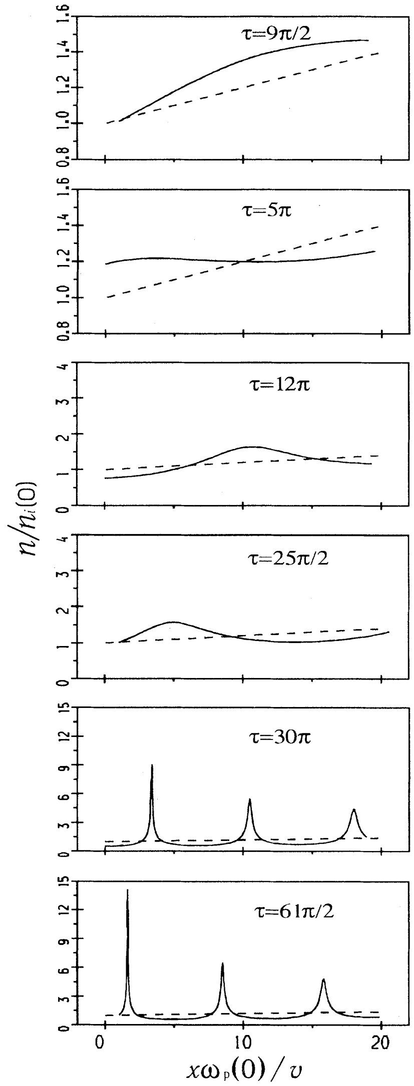

FIG. 3. Space-time evolution of wave density under the conditions of Fig. 2.

$$\frac{(x_{01}-x_{02})\omega_{p1}}{v_1} \approx \left[\frac{v_2}{v_1} \left[1 - \frac{\Delta\omega_p}{\omega_{p1}} \right] - 1 \right] \sin(\omega_{p1} t_c) + \frac{v_2}{v_1} \left[1 - \frac{\Delta\omega_p}{\omega_{p1}} \right] \Delta\omega_p t_c \cos(\omega_{p1} t_c) . \tag{13}$$

Finally, going to the limit $x_{01} - x_{02} = \Delta x_0 \to 0$, we obtain

$$\left[\frac{1}{\omega_p(x_0)} \frac{dv(x_0)}{dx_0} - R \right] \sin(\omega_p t_c) + R\,\omega_p t_c \cos(\omega_p t_c) = 1 . \tag{14}$$

Consider now the solution of Eq. (14) in the two extreme cases, when one item on the left-hand side is much greater than the other. Let $\omega_p t_c$ be small enough so that $R\omega_p t_c \ll 1$. Then the minimum positive value $\omega_p t_c$, satisfying (14), is written in the following way:

$$\omega_p t_c \approx \begin{cases} \arcsin \left[\omega_p \Big/ \left[\dfrac{dv}{dx_0} \right] \right] & \text{at } \dfrac{dv}{dx} > 0 \\ \pi - \arcsin \left[\omega_p \Big/ \left[\dfrac{dv}{dx_0} \right] \right] & \text{at } \dfrac{dv}{dx} < 0 . \end{cases} \tag{15}$$

Intersection of the trajectories is not connected with homogeneity of plasma in this case. As well as in a homogeneous plasma, it is an effect of a big gradient in the initial distribution of velocities in space and takes place until the end of the first period of plasma oscillations. At a small gradient of initial velocity disturbance, when $[1/\omega_p(x_0)]dv(x_0)/dx_0 - R < 1$, the case under consideration does not take place.

On the other hand, if $(1/\omega_p)dv/dx_0 \ll 1$ (i.e., the condition of absence of trajectory intersections is completely fulfilled in a homogeneous plasma), we may neglect the first item in (14). This gives

$$\omega_p t_c \approx \frac{1}{R} , \tag{16}$$

which is consistent with (1). The moment of trajectory intersections is determined by Eq. (16) to within one period of oscillations, which does not introduce a large error, because of $\omega_p t_c \gg 1$. Thus, it follows from the obtained analytical solution that intersections of electron trajectories happen in an inhomogeneous plasma in any case after some time determined by the parameter R. These intersections are produced by the time-increase incoordination of oscillations of the neighboring plasma layers.

B. The number of electrons in density peaks

Since wave breaking in an inhomogeneous plasma takes place at any amplitude the next question is natural: "Why does a space charge of forming bunches not prevent overturning, in contrast to a homogeneous plasma when the amplitude ought to be more than a_{cr} to overcome Coulomb's repulsion?" To answer this question it is necessary to calculate the forming bunch width which is, conditionally, the x distance between intersection points of solid and dashed lines in Fig. 3. The width Δx of the electron layer, situated at arbitrary time t at some of the mentioned intersection points, must satisfy, according to (10), the condition

$$\Delta x \approx \Delta x_0 . \tag{17}$$

Then, differentiating (8) with respect to x_0 and considering, for simplicity, the case when oscillation amplitude does not depend on x_0, i.e., $v(x_0)/\omega_p(x_0)=\mathrm{const}$, we obtain

$$\frac{dx}{dx_0} = 1 + \frac{v}{\omega_p} t \frac{d\omega}{dx_0} \cos[\omega_p(x_0)t] \approx 1 . \tag{18}$$

The relation (18) gives the sequence of values $\omega_p(x_{0,s})$, corresponding to required values x_s,

$$\omega_{p,s} = \frac{(2s+1)\pi/2}{t} , \tag{19}$$

where s is an integer. Later, using the relation between frequencies and coordinates in a linear inhomogeneous plasma,

$$\omega_{p,s+1} = \omega_{p,s} \left\{ 1 + \frac{1}{2} \frac{k(x_{0,s+1} - x_{0,s})}{n(x_{0,s})} \right\} , \tag{20}$$

we find, taking account of (19), the initial distance l between the two nearest layers of interest:

$$l = x_{0,s+1} - x_{0,s} = \frac{2\pi n(x_{0,s})}{t\,\omega_{p,s} k} . \tag{21}$$

It is seen that electron bunch charge, which is directly proportional to l, decreases with time (compare with Fig. 3). Substituting in (21) the value $t=t_c$, we find the minimal value l_c,

$$l_c = \pi \frac{v}{\omega_p} . \tag{22}$$

The distance between the points x_s and x_{s+1} at the same time is equal to $(\pi-2)v/\omega_p$, i.e., ~ 1 on a scale of Fig. 3. Thus, the answer to the formulated question is that with the decrease of oscillation amplitude, a number of bunch electrons decrease to the same degree. The electric field turns out to be always insufficient to prevent intersections of trajectories.

There are a number of plasma systems whose behavior is significantly conditioned by excited Langmuir oscillations. One can name beam-plasma systems; a plasma under intensive laser radiation, which drives plasma waves by stimulated Raman scattering; a powerful high-frequency radio wave, propagating through ionospheric plasma; intensive impulse ion beams, capturing electrons by their Coulomb field; wake-field accelerators, and some others. With large amplitudes of oscillations, when oscillatory velocity v is much more than the thermal one v_e, a cold plasma approach is used, for simplification of analysis. This is done in the present paper as well. However, because formation of density peaks lasts for many periods, thermal motion can still be significant. Conditions under which the considered effect of peak formation takes place even in a warm plasma can be estimated in the following way. According to (22) at time t_c density peaks consistent of electrons which were initially enclosed in space interval $l_c = \pi(v/\omega_p)$. At the same time as t_c, owing to chaotic thermal velocities, every electron sheet spreads by $l_w \approx v_e t_c$. Thermal motion does not destroy the peaks of $l_w < l_c$. Using (16), we obtain the condition of existence of the considered effect in a warm plasma,

$$\frac{v_e}{R} < \pi v .$$

Because R is small, this condition is more rigid than the condition of availability of a cold plasma approach ($v_e < v$). Nevertheless, nearly relativistic ($v \sim 10^{10}$ cm/sec) v can be expected, for example, in wake-field accelerators at large amplitude electric fields. At the same time, v_e is usually 10^8 cm/sec in a discharge plasma. Under these conditions, even relatively weak inhomogeneity of plasma with $R \sim 0.01$ can lead to wave breaking.

C. Accounting for nonlinearity of the original equations

The effect under consideration is connected with the conduct of trajectories over a long time interval. Therefore, the question about the possibility of an appreciable influence of the nonlinear item in Eq. (4) on the obtained solution, despite the item's small value, is logical. One may answer this question if a numerical calculation of trajectories is performed. For this purpose, writing normalized displacements of two electrons under considera-

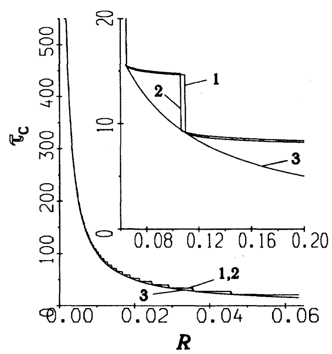

FIG. 4. Time of intersection of neighboring trajectories depending on the parameter R. (1) The result of numerical calculations of trajectories by Eq. (3); (2) numerical solution of Eq. (14); (3) calculation by Eq. (16).

tion as $\delta_1'=(x_1-x_{01})/(v_1/\omega_{p1})$ and $\delta_2'=(x_2-x_{02})/(v_1/\omega_{p1})$ and time as $\tau=\omega_{p1}t_1$, we write the corresponding normalized equations for each of them. An expression is obtained for δ_1',

$$\frac{d^2\delta_1'}{d\tau^2}=-(1+R\delta_1')\delta_1' , \qquad (23)$$

with initial conditions

$$\delta_1'|_{\tau=0}=0 , \quad \left.\frac{d\delta_1'}{d\tau}\right|_{\tau=0}=1 , \qquad (24)$$

and for δ_2',

$$\frac{d^2\delta_2'}{d\tau^2}=-\left[1+R\delta_2'\frac{n_{01}}{n_{02}}\right]\delta_2'\frac{\omega_{p2}^2}{\omega_{p1}^2} , \qquad (25)$$

with initial conditions

$$\delta_2'|_{\tau=0}=0 , \quad \left.\frac{d\delta_2'}{d\tau}\right|_{\tau=0}=\frac{v_2\omega_{p1}}{v_1\omega_{p2}} . \qquad (26)$$

Equations (23) and (25) were solved by the Runge-Kutta method while the trajectory intersection condition was checked at each step,

$$\delta_1'-\delta_2'=\frac{x_{02}-x_{01}}{v_1/\omega_{p1}} . \qquad (27)$$

and, thus, the corresponding moment of time τ_c was found. The dependence of τ_c on the parameter R at $v_2/v_1=1$, $\omega_{p1}/\omega_{p2}=0.999$ is mapped in Fig. 4. The corresponding dependence, obtained by the numerical solution of Eq. (14), is shown in the same figure. Finally, analytical dependence derived from Eq. (16) is depicted. System parameters are identical for all of these curves. It is seen that the analytical solution at small R gives the time of trajectory intersection with good precision, while nonlinearity does not introduce significant changes. This last circumstance permits us to conclude, in particular, that Eq. (16) may be applied, seemingly, when k depends on x, i.e., $n_i(x)$ is a nonlinear function. If it is smooth enough, the cumulative effect of the items with higher powers of δ, which appear in this case in Eq. (4), is also not great.

IV. CONCLUSIONS

One-dimensional Langmuir oscillations in a cold inhomogeneous plasma have been examined. Electron oscillations are excited by an initial push. The intensity distribution of the push as a function of the coordinate is defined without any connection with the inhomogeneity of the ion background. Till the self-intersection of trajectories, every electron moves independently of others and its trajectory is determined by the equation of a nonlinear oscillator with a frequency depending on the coordinate of electron equilibrium. However, if the relative change of ion concentration on the amplitude of the electron displacement (i.e., R, the dimensionless parameter of the inhomogeneity) is much less than 1, trajectory equations are linearized and the process of wave analysis of electron density is facilitated.

In contrast to the case of a homogeneous plasma, unlimited growth of density of the forming electron accumulations takes place at any initial velocity disturbance. The universal equation, valid at any form of spatial distribution of initial push intensity, has been formulated for t_c, the corresponding time of trajectory self-intersections. At small gradients of this initial push, $\omega_p t_c \sim R^{-1}$. The number of electrons making up the condensations is proportional to the amplitude of electron oscillations. Therefore, even at very small amplitudes, the spatial charge is unable to prevent the wave breaking. Taking account of nonlinearity in the trajectory equations according to the numerical calculations does not significantly influence the value t_c found analytically.

ACKNOWLEDGMENT

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